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The effects of a magnetic field in an integrable Heisenberg chain with mixed spins

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Abstract. We discuss some properties of an integrable and conformally invariant Heisenberg chain with alternating spins S and S' ($S' > S$) in the presence of a magnetic field. The zero-field magnetic susceptibility is computed and turns out to be only dependent on the spin S'. For small enough magnetic field, the low-lying excitations are related to the deformed $SU(2)$ parafermionic field theory at levels $2S$ and $2(S' - S)$. Generalizations to these results for other mixed systems are commented on.

1. Introduction

The integrable quantum spin chain has been known as a possible prototype for experimental realizations of conformal field theories [1]. For instance, the underlying conformal field theory of the integrable and isotropic spin-S Heisenberg (XXX-S) model [2, 3] is the $SU(2)$ Wess–Zumino–Witten–Novikov (WZWN) nonlinear σ-model with topological charge $k = 2S$ [4, 5]. Recently some progress has been made on the construction of integrable and conformally invariant magnetic chains combining different kinds of spins [6]. A particularly simple example is an alternating chain composed of spins S and S' acting on odd and even sites of the chain of size L, respectively [6–8]. It has been argued in [8], for a conformally invariant version of this alternating mixed system, that its associated conformal anomaly can be decomposed in terms of two $SU(2)$ WZWN models with topological charge $k = 2S$ and $k = 2(S' - S)$ ($S' > S$).

The main purpose of this paper is to study the properties of this mixed spin chain in the presence of a magnetic field. There are two basic motivations for such an analysis. First, as shown by Affleck [9], besides the specific heat, the zero-temperature magnetic susceptibility plays an important role in determining the class of universality of conformally invariant chains. We find that the zero-field susceptibility is a universal number, when scaled by the sound velocity, proportional to the 'higher' spin S'. Hence, unlike Affleck's discussion [9] for the XXX-S model, this universal number alone will not be a sufficient 'experimental' test in order to determine the spins S and S' of the mixed S–S' chain.

The second purpose is to study the magnetic field as an integrable off-critical perturbation of this mixed system. It turns out that such an analysis will shed some light on the previously mentioned [8] decomposition in terms of two $SU(2)$ WZWN models. By using an approach due to Tsvelick [10] we show that a sufficiently small magnetic field breaks the spectrum into two massless coupled bosonic fields and two massive $SU(2)$ parafermionic field theories at levels $k = 2S$ and $k = 2(S' - S)$. This leads us to conclude that the operator content can be described as a certain combination of bosonic and parafermionic fields.

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This paper is organized as follows. In the next section we review the basic Bethe ansatz properties of this mixed S–S' chain and we compute the zero-field magnetic susceptibility. In section 3 we analyse the excitations appearing in the thermodynamic Bethe ansatz equation in presence of a magnetic field. Section 4 is devoted to the discussion of our results and further generalizations.

2. The mixed S–S' chain and its magnetic susceptibility

We start by briefly reviewing the basic Bethe ansatz results of an integrable and isotropic Heisenberg chain consisting of alternating spins S and S' [6–8, 11]. Here the main purpose is only to consider the properties of conformally invariant mixed S–S' spin chain†. Let us consider such a system consisting of $L/2$ spins S (S') acting on odd (even) sites of the chain in presence of an external magnetic field H ($H > 0$). The spectrum of this model is labelled [6–8, 11] by r disjoint sectors of the eigenvalues of the total spin operator $\tilde{S}^z = \sum_{i=\mathrm{odd}}^{M} S_i + \sum_{i=\mathrm{even}}^{M} S_i'$, where $M = L/2(S+S') - r$. The eigenenergies are parametrized in terms of the rapidities λ_j by

$$E = -\sum_{j=1}^{M} \frac{2S}{(\lambda_j^1)^2 + (S)^2} - \sum_{j=1}^{M} \frac{2S'}{(\lambda_j^1)^2 + (S')^2} - H[L(S+S')/2 - M] \tag{1}$$

where the parameters λ_j satisfy the following Bethe ansatz equation

$$\left(\frac{\lambda_j - \mathrm{i}S}{\lambda_j + \mathrm{i}S}\right)^{L/2} \left(\frac{\lambda_j - \mathrm{i}S'}{\lambda_j + \mathrm{i}S'}\right)^{L/2} = -\prod_{l=1}^{M} \frac{\lambda_j - \lambda_l - \mathrm{i}}{\lambda_j - \lambda_l + \mathrm{i}}. \tag{2}$$

We shall consider the thermodynamic equations associated with the system of equations (1) and (2). In the thermodynamic limit, $L \to \infty$, equations (1) and (2) can be written in terms of the densities of particles $\sigma_n(\lambda)$ and holes $\tilde{\sigma}_n(\lambda)$ for a given distribution of λ_j of n-string type [12]. Following the standard procedure [12, 3] the free energy $F(T, H)$ at temperature T can be expressed in terms of the 'excitation' energies $\epsilon_n(\lambda) = T\ln(\tilde{\sigma}_n(\lambda)/\sigma_n(\lambda))$, $n = 1, 2, \ldots$, by

$$F(T,H)/L = e_\infty - \frac{T}{4\pi}\int_{-\infty}^{\infty} \mathrm{d}\lambda\, p(\lambda)[\ln(1+\mathrm{e}^{\epsilon_{2S}(\lambda)/T}) + \ln(1+\mathrm{e}^{\epsilon_{2S'}(\lambda)/T})] \tag{3}$$

where $e_\infty = -\frac{1}{2}[\psi(S+\frac{1}{2}) + \psi(S'+\frac{1}{2})] - \psi(\frac{1}{2}+S/2+S'/2) + \psi((S'-S+1)/2) + \psi(1/2)$, $\psi(x)$ being the Euler psi function and $p(\lambda) = \pi/\cosh(\pi\lambda)$. The functions $\epsilon_n(\lambda)$ satisfy the so-called thermodynamic Bethe ansatz equation

$$T\ln(1+\mathrm{e}^{\epsilon_n(\lambda)/T}) = -\psi_{n,S}(\lambda) - \psi_{n,S'}(\lambda) + nH + T\sum_{j=1}^{\infty}[A_{n,j} * \ln(1+\mathrm{e}^{-\epsilon_j/T})](\lambda) \tag{4}$$

where $[f * g](x)$ denotes the convolution $(1/2\pi)\int_{-\infty}^{\infty} f(x-y)g(y)\,\mathrm{d}y$. The functions $A_{n,j}(\lambda)$ and $\psi_{n,j}(\lambda)$ can be easily represented in terms of their Fourier transform $A_{n,j}(\omega)$ and $\psi_{n,j}(\omega)$

$$A_{n,j}(\omega) = \coth(|\omega|/2)[\mathrm{e}^{-|n-j||\omega|/2} - \mathrm{e}^{-(n+j)|\omega|/2}] \tag{5}$$

$$\psi_{n,j/2}(\omega) = A_{n,j}(\omega)p(\omega) \tag{6}$$

† For a general discussion on *non-rotational* mixed vertex models leading to mixed spin systems see [6, 7].

where $p(\omega) = 1/2\cosh(\omega/2)$ and the Fourier transform of $f(\lambda)$ is $f(\omega) = (1/2\pi)\int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{-\mathrm{i}\omega x} f(x)$.

From now on we concentrate on the $T = 0$ behaviour of equations (3) and (4) for a small field H ($H > 0$). The first step is to notice that functions $\epsilon_n(\lambda)$ are always positive for $n \neq 2S$ and $2S'$. The change of sign only occurs for $\epsilon_{2S}(\lambda)$ and $\epsilon_{2S'}(\lambda)$ at some point $\lambda = \pm b_1$ and $\lambda = \pm b_2$, respectively. Defining $\epsilon^+_{2S(2S')}(\lambda)(\epsilon^-_{2S(2S')}(\lambda))$ as the positive (negative) part of function $\epsilon_{2S(2S')}(\lambda)$, and taking the $T \to 0$ limit of equation (4) we obtain a new system of equations for $n \neq 2S$ and $2S'$

$$\epsilon_n(\lambda) = -\psi_{n,S}(\lambda) - \psi_{n,S'}(\lambda) + nH - [A_{n,2S} * \epsilon^-_{2S}](\lambda) - [A_{n,2S'} * \epsilon^-_{2S'}](\lambda) \tag{7}$$

and the following matrices system for $n = 2S, 2S'$

$$\begin{pmatrix} \epsilon^+_{2S}(\lambda) \\ \epsilon^+_{2S'}(\lambda) \end{pmatrix} = -\begin{pmatrix} \psi_{2S,S}(\lambda) + \psi_{2S,S'}(\lambda) \\ \psi_{2S',2S}(\lambda) + \psi_{2S',S'}(\lambda) \end{pmatrix} + H\begin{pmatrix} 2S \\ 2S' \end{pmatrix} - [I + K] * \begin{pmatrix} \epsilon^-_{2S} \\ \epsilon^-_{2S'} \end{pmatrix}(\lambda) \tag{8}$$

where I is a 2×2 identity matrix and $K(\lambda)$ is the following matrix

$$K(\lambda) = \begin{pmatrix} A_{2S,2S}(\lambda) - 1 & A_{2S,2S'}(\lambda) \\ A_{2S',2S}(\lambda) & A_{2S',2S'} - 1 \end{pmatrix}. \tag{9}$$

Finally, solving equation (8) in terms of functions $\epsilon_{2S}(\lambda)$ and $\epsilon_{2S'}(\lambda)$ we have

$$\begin{pmatrix} \epsilon_{2S}(\lambda) \\ \epsilon_{2S'}(\lambda) \end{pmatrix} = -\begin{pmatrix} p(\lambda) \\ p(\lambda) \end{pmatrix} + H\begin{pmatrix} 0 \\ 1/2 \end{pmatrix} - [J] * \begin{pmatrix} \epsilon^+_{2S} \\ \epsilon^+_{2S'} \end{pmatrix}(\lambda) \tag{10}$$

where $J(\lambda)$ is the matrix $J(\lambda) = -K * [I + K]^{-1}(\lambda)$.

The next step is to solve equation (10) up to the first non-zero order in the magnetic field H. At this point it is convenient to define the functions $y_1(\lambda) = \epsilon_{2S}(\lambda + b_1)$ and $y_2(\lambda) = \epsilon_{2S'}(\lambda + b_2)$. In terms of these new functions equation (10) can be rewritten as

$$\begin{aligned} y_i(\lambda) = -f_i(\lambda) + Hv_i - \sum_{j=1}^{2}\int_0^\infty \mathrm{d}\lambda'\, J_{i,j}(\lambda - \lambda' + \alpha_{ij})y_j(\lambda') \\ - \sum_{j=1}^{2}\int_0^\infty \mathrm{d}\lambda'\, J_{i,j}(\lambda + \lambda' + 2b_j + \alpha_{ij})y_j(\lambda') \end{aligned} \tag{11}$$

where $f_i(\lambda) = p(\lambda + b_i)$, $v_1 = 0$, $v_2 = 1/2$ and $\alpha_{ij} = b_i - b_j$.

Considering a small magnetic field ($H \ll 1$) one can show, up to order of e^{-b_i} (see, e.g., [3]), that the last term of equation (11) can be neglected and the resultant expression can be cast in form of a system of Wiener–Hopf equations [14]. Defining the functions

$$y_i^{\pm}(\omega) = \frac{1}{2\pi}\int_0^\infty \mathrm{d}\lambda\, y_i(\pm\lambda)\mathrm{e}^{\pm\mathrm{i}\omega\lambda} \tag{12}$$

and by using the standard techniques for solving Wiener–Hopf [14] equations we find (up to order of e^{-b_i})

$$y_i^+(\omega) = -\sum_{j=1}^{2}\frac{[G_+(\omega)G_-(-\mathrm{i}\pi)]_{ij}\mathrm{e}^{-\pi b_j}}{\pi - \mathrm{i}\omega} + \frac{\mathrm{i}H}{2\pi}\sum_{j=1}^{2}\frac{v_j[G_+(\omega)G_-(0)]_{ij}}{\omega + \mathrm{i}0} \tag{13}$$

where functions $G_+(\omega)_{ij}(G_-(\omega)_{ij})$ are analytic on the upper (lower) half complex plane factorizing the kernel $\delta_{i,j} + J_{i,j}(\omega)e^{i\alpha_{ij}\omega}$ as

$$[G_+(\omega)G_-(\omega)]_{ij}^{-1} = \delta_{i,j} + J_{i,j}(\omega)e^{i\alpha_{ij}\omega}. \tag{14}$$

The constants $e^{-\pi b_j}$ are related to the magnetic field through the relation $y_i(\lambda = 0)$, and using equations (12) and (13) we have

$$\sum_{j=1}^{2} G_-(-i\pi)_{ij} e^{-\pi b_j} = \frac{H}{2\pi} \sum_{j=1}^{2} G_-(0)_{ij} v_j. \tag{15}$$

Taking the $T \to 0$ limit in equation (3) we find that the following expression for the free energy ($H \ll 1$)

$$F(0, H)/L = e_\infty - 2\pi \sum_{j=1}^{2} e^{-\pi b_j} y_j^+(i\pi) \tag{16}$$

and finally using the relations (13) and (15) we obtain the result

$$F(0, H)/L = e_\infty - S'H^2/4\pi^2. \tag{17}$$

From this equation we conclude that the zero-field magnetic susceptibility $\chi = -\partial^2 F/\partial^2 H$ satisfies

$$\chi v_s = S'/\pi \tag{18}$$

where $v_s = 2\pi$ is the sound velocity of the mixed S–S' spin chain [6, 8]. Setting $S' = 1$ in equation (17) we recover the result previously obtained in [7]† for the spins $\frac{1}{2}$–1 mixed system. When $S' = S$ (homogeneous system), equation (18) reproduces the early results of [3, 9].

As a consequence of equation (18) the universal number χv_s depends only on the 'higher' spin variable S'. On the other hand, however, this result would be expected if one considered the decomposition mentioned in the introduction and previous conclusions reached by Affleck [9] in the case of the XXX-S chain. Indeed taking into account that this mixed system may be decomposed in terms of two WZWN models with $k = 2S$ and $k = 2(S' - S)$ and that $\chi v_s = k/2\pi$ for the XXX-$k/2$ chain will lead us to conjecture equations (17) and (18) up to order H^2 of the magnetic field (considering that $v_s = 2\pi$ for the mixed $S - S'$ system [6, 8].).

3. The parafermionic and bosonic sectors of the mixed S–S' system

In this section we are going to study the excitation modes appearing in the system of equation (4) for small magnetic field $H \ll 1$. We shall adopt the approach used by Tsvelick

† In this reference the amazing magnetic properties of the non-rotational invariant mixed spin $\frac{1}{2}$–1 model has also been discussed.

[10] in his study of the effects of a magnetic field in the XXX-S chain. We begin by inverting equation (8) in terms of functions $\epsilon^-_{2S(2S')}$

$$\begin{pmatrix} \epsilon^-_{2S}(\lambda) \\ \epsilon^-_{2S'}(\lambda) \end{pmatrix} = -\begin{pmatrix} p(\lambda) \\ p(\lambda) \end{pmatrix} + H\begin{pmatrix} 0 \\ 1/2 \end{pmatrix} - \frac{1}{\det[I+K]}\begin{pmatrix} A_{2S',2S'}(\lambda) & -A_{2S',2S}(\lambda) \\ -A_{2S,2S'}(\lambda) & A_{2S,2S} \end{pmatrix} * \begin{pmatrix} \epsilon^+_{2S} \\ \epsilon^+_{2S'} \end{pmatrix}(\lambda). \tag{19}$$

Substituting this equation back into equation (7) it follows that

$$\epsilon_n(\lambda) = [B^S_{n,2S-1} * p * \epsilon^+_{2S}](\lambda) \qquad n = 1, 2, \ldots, 2S-1 \tag{20}$$

$$\epsilon_{n+2S}(\lambda) = [B^{S'-S}_{n,1} * p * \epsilon^+_{2S}](\lambda) + [B^{S'-S}_{n,2(S'-S)-1} * p * \epsilon^+_{2S'}](\lambda) \qquad n = 1, 2, \ldots, 2(S'-S)-1 \tag{21}$$

$$\epsilon_{n+2S'}(\lambda) = nH + [A_{n,1} * p * \epsilon^+_{2S'}](\lambda) \qquad n = 1, 2, \ldots \tag{22}$$

where the Fourier transform of the matrix elements of $B^N_{n,j}(\omega)$ satisfies the following property

$$[B^N]^{-1}_{n,j} = \delta_{i,j} - p(\omega)(\delta_{i,j+1} + \delta_{i,j-1}) \qquad i, j = 1, 2, \ldots, 2N-1. \tag{23}$$

As has been mentioned in section 2 the functions $\epsilon_n(\lambda) \geqslant 0$ for $n \neq 2S, 2S'$ (see equation (20)). The modes $n = 2S, 2S'$ can be considered as the ground state of this system provided that the magnetic field is small enough in order to avoid crossings (for all λ) between these modes and those of $n \neq 2S, 2S'$. In order to classify the 'positive' modes ($n \neq 2S, 2S'$) it is important to estimate the 'mass gap' $\epsilon_n(0)$ around the origin in terms of the magnetic field. From equation (22) it is straightforward to conclude that $\epsilon_{n+2S'}(0) \sim nH, n = 1, 2, \ldots$. On the other hand, for $|\lambda| \ll \min(b_1, b_2)$, it follows from equations (20) and (21) and by some manipulations of functions $B^N_{n,j}(\lambda)$ that

$$\epsilon_n(\lambda) = \frac{1}{S} m_1(S) \sin[\pi n/2S] \cosh(\lambda\pi/S) \qquad n = 1, 2, \ldots, 2S-1 \tag{24}$$

$$\epsilon_{n+2S}(\lambda) = \frac{1}{S'-S}[m_1(S'-S) + m_2(S'-S)] \sin[\pi n/2(S'-S)] \cosh(\lambda\pi/(S'-S)) \qquad n = 1, 2, \ldots, 2(S'-S)-1 \tag{25}$$

where

$$m_1(N) = 2\int_{b_1}^{\infty} d\lambda' \, e^{-\lambda'\pi/N} \epsilon_{2S}(\lambda') \qquad \text{and} \qquad m_2(N) = 2\int_{b_2}^{\infty} d\lambda' \, e^{-\lambda'\pi/N} \epsilon_{2S'}(\lambda').$$

From equations (10), (15) and (19) we notice that $e^{-\pi b_i} \sim H$ and that $\epsilon_{2S}(\lambda \to \infty) \sim \epsilon_{2S'}(\lambda \to \infty) \sim H$. Substituting these relations in equations (24) and (25) one is able to estimate the following 'mass gaps'

$$\epsilon_n(0) \sim \sin(\pi n/2S) H^{1+1/S} \qquad n = 1, 2, \ldots, 2S-1 \tag{26}$$

$$\epsilon_{n+2S} \sim \sin(\pi n/2(S'-S)) H^{1+1/(S'-S)} \qquad n = 1, 2, \ldots, 2(S'-S)-1. \tag{27}$$

Considering that we have assumed $H \ll 1$ and from the power of H of equations (26) and (27) we conclude that the modes $\epsilon_{n+2S'}(\lambda)$, $n = 1, 2, \ldots$, become the heaviest massive excitation of the system. Hence, the low-lying massive excitation corresponds to the lightest mass gaps $\epsilon_n(\lambda), n = 1, 2, \ldots, 2S - 1, 2S + 1, \ldots, 2S' - 1$. These are, indeed, the non-trivial modes appearing if one considers the magnetic field as an off-critical perturbation of the conformal point. From the standard results of perturbation theory one expects that the mass gaps scale as $H^{1/(1-\Delta_\phi)}$, where Δ_ϕ is the conformal dimension of the perturbating field. Comparing this last relation with equations (26) and (27) we notice that the low-lying massive spectrum can be generated by the direct sum of two operators with conformal dimension $\Delta_{\phi_1} = 1/(1+S)$ and $\Delta_{\phi_2} = 1/(1 + S' - S)$. It turns out that such conformal dimension are those associated to the thermal operator of a $Z(2S)$ and $Z(2(S' - S))$ parafermionic theory [15], respectively. This suggests that the low-lying massive excitations are, in some sense, related to the deformed parafermionic field theory by its 'first' energy operator. In order to investigate this fact, let us rewrite the thermodynamic Bethe ansatz equations in terms of the densities of particles ($\sigma_n(\lambda)$ and holes ($\tilde{\sigma}_n(\lambda)$). Taking into account equations (2) and (4), densities $\sigma_n(\lambda)$ and $\tilde{\sigma}_n(\lambda)$ are constrained by the relation

$$\sigma_n(\lambda) + \tilde{\sigma}_n(\lambda) = p * [\tilde{\sigma}_{n+1} + \tilde{\sigma}_{n-1}(\lambda)](\lambda) + \frac{p(\lambda)}{4\pi}(\delta_{n,2S} + \delta_{n,2S'}). \tag{28}$$

Since we are interested in the low-lying modes we just have to solve this equation for $n = 1, 2, \ldots, 2S - 1, 2S + 1, \ldots, 2S' - 1$ in terms of densities $\tilde{\sigma}_{2S}(\lambda)$ and $\tilde{\sigma}_{2S'}(\lambda)$. The final result is written as

$$\tilde{\sigma}_n(\lambda) + \sum_{j=1}^{2S-1} [B^S_{n,j} * \sigma_j](\lambda)[B^S_{n,2S-1} * p * \tilde{\sigma}_{2S}](\lambda) \qquad n = 1, 2, \ldots, 2S - 1 \tag{29}$$

$$\tilde{\sigma}_{n+2S}(\lambda) + \sum_{j=1}^{2(S'-S)-1} [B^{S'-S}_{n,j} * \sigma_{j+2S}](\lambda) = [B^{S'-S}_{n,1} * p * \tilde{\sigma}_{2S})](\lambda) + [B^{S'-S}_{n,2(S'-S,-1)} * p * \tilde{\sigma}_{2S'}](\lambda)$$
$$n = 1, 2, \ldots, 2(S' - S) - 1. \tag{30}$$

Considering the similar analysis made for the functions $\epsilon_n(\lambda)$, one notices that the asymptotic limit ($H \ll 1$) of the right-hand side of equations (29) and (30) is practically parallel with those performed in equations (20) and (21). Indeed, taking into account the approximations made in equations (24) and (25), we can show that such equations become

$$\tilde{\sigma}_n(\lambda) + \sum_{j=1}^{2S-1} [B^S_{n,j} * \sigma_j](\lambda) = m_1(S)\sin(\pi n/2S)\cosh(\pi\lambda/S) \qquad n = 1, 2, \ldots, 2S - 1 \tag{31}$$

$$\tilde{\sigma}_{n+2S}(\lambda) + \sum_{j=1}^{2(S'-S)-1} [B^{S'-S}_{n,j} * \sigma_{j+2S}](\lambda)$$
$$= [m_1(S' - S) + m_2(S' - S)]\sin(\pi n/2(S' - S))\cosh(\pi\lambda/(S' - S))$$
$$n = 1, 2, \ldots, 2(S' - S) - 1. \tag{32}$$

As has been argued by Tsvelick [10], equations (31) and (32) define the thermodynamic Bethe ansatz equation of the $Z(2S)$ and $Z(2(S' - S))$ parafermionic field theories perturbed by the respective energy operator with conformal dimension $\Delta = 1/(1+S)$ and $\Delta = 1/(1+S'-S)$. Nevertheless, by studying the ultraviolet limit of equations (31) and (32) [16] we conclude that the central charges are indeed those of these parafermionic models, namely $c = 3S/(S+1)$ and $c = 3(S'-S)/(1+S'-S)$. In order to complete our study it remains to analyse the finite-size corrections of the 'vacuum' modes $n = 2S$ and $n = 2S'$. In this case we have used the analytical method introduced in [17]. The computation is summarized in the appendix and in what follows we present the main result. In the ultraviolet limit these modes contribute to the free energy with a conformal anomaly $c = 2$. Moreover, we have verified that the excitations over the sea of $n = 2S$ and $n = 2S'$ strings are coupled through the term $n_1 n_2/2S'$, where n_1 and n_2 are integer numbers representing the 'spin wave' behaviour of two $c = 1$ bosonic fields. We believe that all these results strongly support the conjecture of [8] that the spectrum of this mixed spin system can be represented in terms of two $SU(2)$ WZWN field theory at levels $k = 2S$ and $k = 2(S'-S)$.

4. Conclusion

In this work we have considered the effects of a magnetic field on an integrable system of alternating spins S and S'. Although the specific heat (C) depends on both S and S' [7, 8] we have shown that the zero-field magnetic susceptibility depends only on the higher spin S'. In this sense, besides the magnetic susceptibility, it seems important to us to define for this model the universal number determined by the Wilson ratio $W = \lim_{T\to 0} C/T\chi = \pi^2[2S(S'-S)+S']/S'(S+1)(S'-S+1)$. Hopefully, this number may be compared with those obtained from a phenomenological Fermi liquid theory.

For a enough small magnetic field, our analysis has revealed that the low-lying excitations are related to the deformed $SU(2)_{2S}$ and $SU(2)_{2(S'-S)}$ parafermionic field theory. This fact together with our computation of the ultraviolet behaviour of the ground state gives an extra support to the conjecture that the operator content of this mixed models can be described by a combination of the parafermionic and bosonic fields.

A natural extension of our results is to consider an alternating $GL(N)$ mixed model at the order k and k' of the symmetric representation constructed in [11]. Let us announce the following results for these mixed $GL(N)$ models [18]. In presence of a rank homogeneous magnetic field, the low-lying excitations will be described by two $SU(N)$ parafermionic theories at levels k and $k'-k$ ($k' > k$) perturbed by the 'energy operator' with conformal dimension $\Delta = N/(k+N)$ and $\Delta = N/(k'-k+N)$, respectively. The bosonic degrees of freedom will contribute to the Casimir energy with a conformal anomaly $c = 2(N-1)$. Moreover, the rank homogeneous combination of the zero-field susceptibility scaled by the sound velocity satisfies the relation $\chi v_s = k'N(N^2-1)/12\pi$.

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Appendix

In this appendix we summarize the finite-size corrections for the ground state of $n = 2S$ and $2S'$ strings. We shall use the method developed in [17] (for a recent review see [19]), which basic idea consists in the computation of the difference of the energy and the root density in a finite system of size L from their thermodynamic limit ($L \to \infty$ values). Following [17] we have the energy

$$E/L - e_\infty = -\pi \int_{-\infty}^{\infty} \sigma_\infty^1(\mu) S^1(\mu)\,\mathrm{d}\mu - \pi \int_{-\infty}^{\infty} \sigma_\infty^2(\mu) S^2(\mu)\,\mathrm{d}\mu \tag{A.1}$$

where $\sigma_\infty^1(\mu) = \sigma_\infty^2(\mu) = 1/2\cosh(\pi\mu)$ are the bulk densities of the strings of size $n = 2S$ and $2S'$, respectively. The functions $S^i(\mu)$ are defined by

$$S^i(\mu) = \frac{1}{L}\sum_j \delta(\lambda_j - \mu) - \sigma_L^i(\mu) \tag{A.2}$$

where $\sigma_L^1(\mu)(\sigma_L^2(\mu))$ is the 'finite' root density of strings $n = 2S$ $(2S')$. Analogously, the root densities satisfy the following equation

$$\begin{pmatrix} \sigma_L^1(\lambda) - \sigma_\infty^1(\lambda) \\ \sigma_L^2(\lambda) - \sigma_\infty^2(\lambda) \end{pmatrix} = \frac{1}{\det[I+K]} \begin{pmatrix} A_{2S',2S'} - 1(\lambda) & -A_{2S',2S}(\lambda) \\ -A_{2S,2S'}(\lambda) & A_{2S,2S} - 1 \end{pmatrix} * \begin{pmatrix} S^1 \\ S^2 \end{pmatrix}(\lambda). \tag{A.3}$$

For large L, according to the method of [17, 19], equation (A.3) can be transformed in a Wiener–Hopf equation in the variable $\Sigma^i(\lambda) = \sigma_L^i(\lambda + \Lambda_i)$, where Λ_i is the largest root magnitude determined by the condition

$$\int_{\Lambda_i}^{\infty} \sigma_L^i(\lambda)\,\mathrm{d}\lambda = \frac{1}{L} + \frac{2r_i}{L} \tag{A.4}$$

where $r_1 = n_1 + Sn_2/S'$, $r_2 = n_1 + n_2$, and $n_1(n_2)$ is an integer number characterizing the 'spin wave' excitations over the ground state of $2S(2S')$ string. Finally, by solving the Wiener–Hopf system of equations for $\Sigma^i(\lambda)$ (the solution is fairly parallel with that of equation (11) of section 2) and taking into account the condition (A.4) we obtain

$$\frac{E}{L} - e_\infty = \frac{4\pi^2}{L^2}\left(-\frac{1}{6} + \frac{n_1^2}{4S} + \frac{n_2^2}{4S'} + \frac{n_1 n_2}{2S'}\right) \tag{A.5}$$

Considering the predictions of conformal invariance for a finite system of size L [4, 20, 21] and that the sound velocity is $v_s = 2\pi$ we find that the central charge is $c = 2$ and the conformal dimension associated with the excitations n_1, n_2 is

$$X_{n_1,n_2} = \left(\frac{n_1^2}{4S} + \frac{n_2^2}{4S'} + \frac{n_1 n_2}{2S'}\right).$$

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